On the Measured Equation of Invariance for an Electrically Large Cylinder

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Abstract—**By considering the electromagnetic scattering of a circular conducting cylinder, the accuracy of the measured equation of invariance (MEI) is examined analytically. Using the asymptotic expansions of the Hankel's functions of second kind, $H_n^{(2)}(k\rho)$ as measuring functions, the case of an electrically large cylinder is studied. It is shown that the MEI coefficients are invariant to the field of excitation.**

I. Introduction

USING the measured equation of invariance (MEI) postulated by Mei in 1992 [1], we can construct a conformal mesh and terminate it very close to the physical boundary of a scatterer, usually two discretization steps, when an electromagnetic field scattering problem is solved by the frequency-domain finite-difference approach. Although no rigorous proof is available in the literature, the MEI has been applied successfully to solve several scattering and antenna problems [2]. Unfortunately, Jevtic and Lee [3], [4] have attempted to disprove the third postulate: the MEI is invariant to the field of excitation. Their analysis, however, has many obvious defects, as explained in [5], [6]. In addition, they did not realize that the asymptotic expansion of the Hankel's function $H_n^{(2)}(k\rho)$ is limited to $n \ll k\rho$ [4]. In this paper, the accuracy of the MEI coefficients is studied analytically. The case of electromagnetic scattering of an electrically large circular conducting cylinder is considered.

II. Analysis

As shown in Fig. 1, a circular conducting cylinder of radius a illuminated by a TM polarized wave $E_z^{\rm inc}$ is investigated. Two layers of regular finite difference mesh are constructed such that the radial and angular spacings between neighboring nodal points are $\Delta\rho$ and $\Delta\phi$, respectively, with $\Delta\rho = \rho\Delta\phi$. The four neighboring nodes, P_1, P_2, P_3, and P_4 are considered. The radius of the outer mesh boundary is $\rho_1 = a + 2\Delta\rho$. For this four-point case, we have the following MEI:

$$\sum_{p=1}^{4} a_p E_{zp} = 0 \tag{1}$$

where E_{zp} is the scattered electric field at node p.

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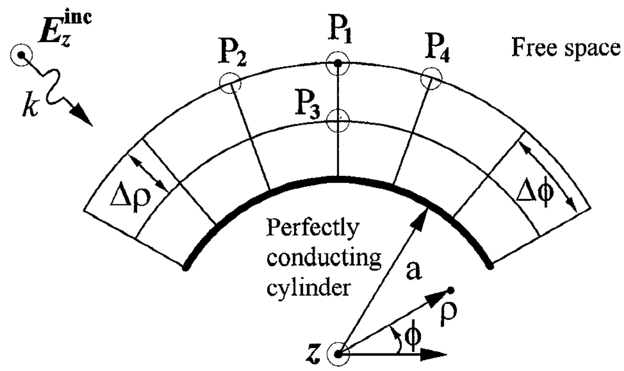

Fig. 1. Geometry for the analysis.

To determine the MEI coefficients in (1) analytically, the following metrons of surface current densities are selected:

$$J_z = e^{jn\phi}, \quad J_z = e^{-jn\phi} \quad J_z = e^{jm\phi}, \quad J_z = e^{-jm\phi} \tag{2}$$

where n, m are arbitrary integers. It should be noted that these metrons can be realized by appropriate incident fields for the special case of circular cylinder. Subsituting the corresponding measuring functions of the metrons into (1), the following four measures are derived:

$$1 = a_2 e^{jn\Delta\phi} + a_3 h_n + a_4 e^{-jn\Delta\phi} \tag{3a}$$
$$1 = a_2 e^{-jn\Delta\phi} + a_3 h_n + a_4 e^{jn\Delta\phi} \tag{3b}$$
$$1 = a_2 e^{jm\Delta\phi} + a_3 h_m + a_4 e^{-jm\Delta\phi} \tag{3c}$$
$$1 = a_2 e^{-jm\Delta\phi} + a_3 h_m + a_4 e^{jm\Delta\phi} \tag{3d}$$

where $h_{n,m} = H_{n,m}^{(2)}(k\rho_1 - k\Delta\rho)/H_{n,m}^{(2)}(k\rho_1), k$ is the wavenumber of the medium. In deriving (3), $a_1 = -1$ is chosen. Considering the subtractions (3a) − (3b) and (3c) − (3d), we find $a_2 = a_4$. Solving (3a) + (3b) and (3c) + (3d), we obtain the following solution:

$$a_2 = a_4 = \frac{(h_m - h_n)}{2(h_m \cos n\Delta\phi - h_n \cos m\Delta\phi)} \tag{4}$$

$$a_3 = \frac{(\cos n\Delta\phi - \cos m\Delta\phi)}{(h_m \cos n\Delta\phi - h_n \cos m\Delta\phi)}. \tag{5}$$

To simplify (4) and (5), we take the following approximations:

$$h_n = 1 - k\Delta\rho \frac{H_n^{(2)'}}{H_n^{(2)}} + \frac{1}{2}k^2\Delta\rho^2 \frac{H_n^{(2)''}}{H_n^{(2)}} - \frac{1}{6}k^3\Delta\rho^3$$

$$a_2 = a_4 = \frac{1}{2}\frac{\left[1+j\frac{1}{k\rho_1}\right] + jk\Delta\rho\left[1-j\frac{1}{2k\rho_1}\right] - \frac{k^2\Delta\rho^2}{2}\left[1-j\frac{2}{k\rho_1}\right] - j\frac{k^3\Delta\rho^3}{6}\left[1-j\frac{7}{2k\rho_1}\right]}{\left[1+j\frac{1}{k\rho_1}\right] + j2k\Delta\rho\left[1-j\frac{1}{4k\rho_1}\right] - \frac{3k^2\Delta\rho^2}{2}\left[1-j\frac{1}{k\rho_1}\right] - j\frac{2k^3\Delta\rho^3}{3}\left[1-j\frac{13}{8k\rho_1}\right]} + O(k^4\Delta\rho^4) \quad (14)$$

$$a_3 = \frac{jk\Delta\rho}{\left[1+j\frac{1}{k\rho_1}\right] + j2k\Delta\rho\left[1-j\frac{1}{4k\rho_1}\right] - \frac{3k^2\Delta\rho^2}{2}\left[1-j\frac{1}{k\rho_1}\right] - j\frac{2k^3\Delta\rho^3}{3}\left[1-j\frac{13}{8k\rho_1}\right]} + O(k^4\Delta\rho^4). \quad (15)$$

$$\cdot \frac{H_n^{(2)'''}}{H_n^{(2)}} + \frac{1}{24}k^4\Delta\rho^4\frac{H_n^{(2)''''}}{H_n^{(2)}} \quad (6)$$

$$\cos n\Delta\phi = 1 - \frac{1}{2}n^2\Delta\phi^2 + \frac{1}{24}n^4\Delta\phi^4. \quad (7)$$

Using the asymptotic expansion for $H_n^{(2)}(k\rho)$ available in [7], we have

$$\frac{H_n^{(2)'}}{H_n^{(2)}} = -j\left[1 - j\frac{1}{2k\rho} - \frac{4n^2-1}{8k^2\rho^2} - j\frac{4n^2-1}{8k^3\rho^3}\right] \quad (8)$$

$$\frac{H_n^{(2)''}}{H_n^{(2)}} = -\left[1 - j\frac{1}{k\rho} - \frac{2n^2+1}{2k^2\rho^2} + j\frac{4n^2-1}{8k^3\rho^3}\right] \quad (9)$$

$$\frac{H_n^{(2)'''}}{H_n^{(2)}} = j\left[1 - j\frac{3}{2k\rho} - \frac{12n^2+15}{8k^2\rho^2} + j\frac{24n^2+9}{8k^3\rho^3}\right] \quad (10)$$

$$\frac{H_n^{(2)''''}}{H_n^{(2)}} = \left[1 - j\frac{2}{k\rho} - \frac{2n^2+4}{k^2\rho^2} + j\frac{28n^2+23}{4k^3\rho^3}\right]. \quad (11)$$

It is important to note that the above asymptotic expansions are limited to $n \ll k\rho$. All the terms of orders higher than $(k\rho)^{-3}$ in (8)–(11) have been truncated. Using these equations, we obtain the following approximations:

$$h_m - h_n = \frac{(n^2-m^2)k\Delta\rho}{2k^2\rho_1^2}\left[j\left(1+j\frac{1}{k\rho_1}\right) - k\Delta\rho\left(1-j\frac{1}{2k\rho_1}\right) - j\frac{k^2\Delta\rho^2}{2}\left(1-j\frac{2}{k\rho_1}\right) + \frac{k^3\Delta\rho^3}{6}\left(1-j\frac{7}{2k\rho_1}\right) + O(k^4\Delta\rho^4)\right] \quad (12)$$

$$h_m(\cos n\Delta\phi - 1) - h_n(\cos m\Delta\phi - 1) = \frac{(n^2-m^2)k\Delta\rho}{2k^2\rho_1^2}\left[-k\Delta\rho - jk^2\Delta\rho^2\left(1-j\frac{1}{2k\rho_1} + \frac{1}{8k^2\rho_1^2} + j\frac{1}{8k^3\rho_1^3}\right) - \frac{k^3\Delta\rho^3}{2}\left(1-j\frac{1}{k\rho_1} + \frac{n^2+m^2-3}{6k^2\rho_1^2} - j\frac{1}{8k^3\rho_1^3}\right) + O(k^4\Delta\rho^4)\right] \quad (13)$$

Subsituting (7), (12), and (13) into (4) and (5), we have (14) and (15), shown at the top of the page.

In (12)–(15), terms of order higher than $k^3\Delta\rho^3$ have not been included. We consider this approximation more than sufficient as the conventional finite-difference equation is only accurate up to the order of $k^2\Delta\rho^2$. It is observed in (14) and (15) that the MEI coefficients: a_2, a_3, and a_4, are invariant to the field of excitation.

In the limiting case when $\Delta\rho$ tends to zero, (1) converges to a differential equation

$$0 = \left(j + \frac{1}{2k\rho} + \frac{j}{8k^2\rho^2} - \frac{1}{8k^3\rho^3}\right)E + \frac{\partial E}{\partial k\rho} + \left(\frac{j}{2k^2\rho^2} - \frac{1}{2k^3\rho^3}\right)\frac{\partial^2 E}{\partial\phi^2} \quad (16)$$

which is also invariant to the field of excitation. It should be noted that in deriving (16), terms of order higher than $k^2\Delta\rho^2$ are not required.

We may also be interested to investigate the invariance behavior of the 6-point measured equation as considered in [3] and [4]. The derivations for this case are much more lengthly. For brevity, we present only the result when $\Delta\rho \to 0$. Using the metrons $J_z = 1, e^{jn\phi}, e^{-jn\phi}, e^{jm\phi}, e^{-jm\phi}$, we find the following equivalent differential equation:

$$0 = \left(j + \frac{1}{2k\rho} + \frac{j}{8k^2\rho^2} - \frac{1}{8k^3\rho^3}\right)E + \frac{\partial E}{\partial k\rho} + \left(\frac{j3}{4k^2\rho^2} - \frac{1}{2k^3\rho^3}\right)\frac{\partial^2 E}{\partial\phi^2} + \left(\frac{1}{4k^2\rho^2} + \frac{j7}{8k^3\rho^3}\right)\cdot\frac{\partial^3 E}{\partial k\rho\partial\phi^2} \quad (17)$$

which is also independent of m and n. In [4], two different differential equations are obtained. One is similar to (17), the other one may be wrongly obtained by choosing $n = k\rho_1$. The fact that the asymptotic expansion for $H_n^{(2)}(k\rho_1)$ is only limited to $n \ll k\rho_1$ has not been realized.

III. Conclusion

We have shown analytically that the MEI coefficients for an electrically large cylindrical mesh are invariant to the field of excitation. In the limiting case when $\Delta\rho$ tends to zero, the measured equation is also invariant to the field of excitation.

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